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TAP complexity, the cavity method and supersymmetry

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Abstract

We compute the Bray and Moore (BM) TAP complexity for the Sherrington–Kirkpatrick model through the cavity method, showing that some essential modifications are needed with respect to the standard formulation of the method. This allows us to understand various features recently discovered and to unveil at last the physical meaning of the parameters of the BM theory. We also reconsider the supersymmetric (SUSY) formulation of the problem finding that the BM solution satisfies some proper SUSY Ward identities that are different from the standard ones. The SUSY relationships encode the physical meaning of the parameters obtained through the cavity method. The problem of the vanishing prefactor is addressed, showing how it can be avoided.

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1. Introduction

One of the most intriguing features of spin glasses and more widely of disordered systems is the fact that the disordered nature of the Hamiltonian prevents one from easily characterizing the ground state of the system or the equilibrium state at finite temperature [1]. Actually it is largely believed that in mean field models there are an exponential number of metastable equilibrium states, those with lower free energy representing the many possible equilibrium states. It is also believed that the peculiar dynamical features of these systems, i.e. their extremely slow dynamics, are somehow related to this high number of states and the consequent roughness of the free energy landscape. Although this connection is far from being clearly established [2], a complete characterization of the metastable states of these system is of great theoretical importance. The problem has been around for more than 20 years and recent progress in the past two years has given a considerable boost in its understanding. The study of paradigmatic models like the Sherrington–Kirkpatrick spin-glass model is the basis for understanding more specific situations in which it is believed that disorder leads to the presence of many equilibrium states. Such cases include e.g. the number of fixed points of the belief-propagation algorithm [3] which is applied to a large variety of practical problems (e.g., error-correcting codes) and the number of solutions of some satisfiability problems [9, 8]. All these different problems share with the SK model the fact that the states can be characterized as solutions of certain disordered equations (e.g. the Thouless–Anderson–Palmer (TAP) equations for the SK model) that can be obtained extremizing a free energy functional [3].

The cavity method was introduced nearly 20 years ago in [5] in order to recover the Parisi solution of the Sherrington–Kirkpatrick (SK) spin-glass model without the replica method and starting explicitly from the hypotheses encoded in his replica-symmetry breaking (RSB) ansatz [1]. In recent years this method has proven to be a powerful tool to investigate disordered systems with tree-like structure and optimization problems (see [6–9] and references therein). In particular it has been possible to obtain a non-perturbative equilibrium solution of the spin-glass on the Bethe lattices at the 1RSB level [6]. The method has been also used to compute the complexity, i.e. the logarithm of the number of metastable states with free energy higher than the equilibrium one [7]. It is natural to expect that in the limit of high connectivity this model should become identical to the SK model and the corresponding complexity curve should be equal to the number of stable solutions of the Thouless–Anderson–Palmer (TAP) equations [10]. However, the resulting complexity turns out to be different from the TAP complexity computed more than 20 years ago by Bray and Moore (BM) in [4] and instead to be equal to another complexity curve yielded by the so-called supersymmetric (SUSY) solution [11, 12, 14, 15]. This has motivated a renewed interest [12–15] in the problem of the TAP complexity starting from the observation that the BM solution violates a supersymmetry (SUSY) [16–19] as first noted in [20]. Furthermore after the observation [15] that the SUSY solution is unstable in the SK model (except at the lower band edge) both at the annealed and at the quenched level, the BM solutions have remained the unique candidate to describe the TAP complexity and it has become mandatory to solve the problems posed by the SUSY violation; as we shall briefly report in the following pages some important results in this direction have been obtained in [21] and rigorously confirmed in [22]. In this paper we shall address the problem of the formulation of the BM solution within the cavity method obtaining as a byproduct to clarify various issues connected with the SUSY violation and eventually to obtain the physical meaning of the parameters λ and Δ of the theory. We want to count the number of solutions of the TAP equations $\partial_i F_{\mathrm{TAP}}(m) = 0$, where $F_{\mathrm{TAP}}(m)$ is a given model-dependent TAP free energy. The BM theory considers the SK model whose free energy is

$$F_{\mathrm{TAP}}(m) = -\frac{1}{2}\sum_{ij} J_{ij} m_i m_j + \frac{1}{\beta}\sum_i \phi_0(q, m_i), \tag{1}$$

with,

$$\phi_0(q, m) = \frac{1}{2}(1+m)\log\left[\frac{1}{2}(1+m)\right] + \frac{1}{2}(1-m)\log\left[\frac{1}{2}(1-m)\right] - \frac{\beta^2}{4}(1-q)^2. \tag{2}$$

In particular we want to compute the density of TAP solutions of a given free energy,

$$\rho(f) = \sum_{\alpha=1}^{\mathcal{N}} \delta[F_{\mathrm{TAP}}(m^\alpha) - Nf]. \tag{3}$$

In the previous expression TAP solutions are labelled by the index α, and $\{m^\alpha\}$ indicates the corresponding set of local magnetizations. The density can be expressed as an integral over the whole m-space of a delta function of the TAP equations:

$$\rho(f) = \sum_{\alpha=1}^{\mathcal{N}} \int \prod_i \mathrm{d}m_i \, \delta\big(m_i - m_i^\alpha\big)\delta[F_{\mathrm{TAP}}(m^\alpha) - Nf]$$
$$= \int \prod_i \mathrm{d}m_i \, \delta(\partial_i F_{\mathrm{TAP}}(m))|\det(\partial_i\partial_j F_{\mathrm{TAP}}(m))|\delta[F_{\mathrm{TAP}}(m) - Nf]. \qquad (4)$$

In order to use an exponential representation of the determinant its modulus is dropped. This is a crucial step that corresponds to count each solution with a weight proportional to the sign of its determinant. The resulting object is not the number of solutions with free energy f but rather the number of solutions with positive determinant minus the number of solutions with negative determinant. Once an exponential representation is obtained the total action can be expressed by means of the even and odd Hubbard–Stratonovich transformations (see, e.g., [22] for the details) as an integral over the variable u and eight macroscopic bosonic and fermionic variables $\Theta \equiv \{r, t, q, \lambda, \overline{\rho}, \rho, \overline{\mu}, \mu\}$

$$\rho_s = \int \mathrm{d}u \, \mathrm{d}\Theta \exp[N\Sigma_1 + N\Sigma_2]. \qquad (5)$$

The BM action equation (15) of [4] is obtained from the previous expression by setting the fermionic variables to zero and changing variables from $\{r, t, q, \lambda\}$ to $\{B, \Delta, q, \lambda\}$. The macroscopic action can be evaluated by the saddle point method in the thermodynamic limit. The BM solution of the resulting saddle point equations yields a bell-shaped complexity curve $\Sigma(f)$ describing solutions of the TAP equations with a given free energy. The above action possesses a SUSY that is violated by the BM solution [20]. We refer the reader to [12, 14, 20] for the discussion of the SUSY of the problem and in particular to [22] whose notation will be adopted in the discussion of section 3. This somewhat exotic symmetry is related to important physical features of the problem. The first is the Morse theorem that states that the number of solutions of the TAP equations with positive Hessian (e.g. minima) minus the number of solutions with negative Hessian (the saddles) is a topological invariant equal to one. It is the generalization of the one-dimensional result that is easily understood. In particular if we find an exponential number of minima we must find also an exponential number of saddles. Before the investigations of recent months it was believed that the BM solution described minima of the TAP free energy, therefore an unavoidable question was: where are the saddles? Furthermore, given that those supposed minima of the TAP free energy had a positive definite Hessian, it should be possible to continue each one of them upon continuously changing the external parameters of the TAP equations such as temperature or magnetic field and also upon adding a spin to the system of N spins. The possibility of continuing the solutions poses two problems. The first problem, noted in [14], is that starting from the hypothesis that the TAP solutions can be continued, one can obtain some relations that are identical to the Ward identities yielded by the SUSY of the problem; thus there is a contradiction between the possibility of continuing the TAP solutions and the fact that the BM solution describing them is not SUSY and does not satisfy the corresponding Ward identities. The second problem is that since the solutions can be continued upon changing the external parameters they cannot disappear, therefore their number must remain the same at all temperatures and magnetic fields, but this is in contradiction with the fact that the BM complexity varies with temperature and field. On the other hand, *if the complexity changes with the temperature and the magnetic field, the relevant exponential number of solutions at given values of the external parameters must disappear upon a small change of them and therefore the Hessian of these solutions must have at least a zero eigenvalue*. This is a crucial argument which was missed up to now in the discussion.

These problems of the BM solution have been solved by reconsidering the TAP spectrum. In [21] it was pointed out that the spectrum contains an isolated eigenvalue besides the continuous positive band and it was checked numerically that this eigenvalue vanishes on the TAP solutions described by the BM solution, this result has been proven rigorously in [22] and it is precisely a consequence of the SUSY violation. Following [21, 22] we recall that the appearance of the isolated eigenvalue is connected to the fact that the Hessian X^{-1} can be written in the form $A = B + P$ where P is a projector. Given two symmetric matrices A and B that differ by a projector $P = |\alpha\rangle\langle\alpha|$ we have

$$A = B + P \longrightarrow \det A = (1 + \langle\alpha|B^{-1}|\alpha\rangle) \det B, \tag{6}$$

that is the determinant of the total Hessian A is the product of the determinant of B times a factor depending on B and on the projection vector. The matrix B is essentially the matrix of the interactions and it can be studied through standard random matrix theory [23–26]. It turns out to have always a positive spectrum with a continuous band of eigenvalues whose lower band edge extends down to zero if the following quantity is zero:

$$x_p = 1 - \beta^2 \sum_i \left(1 - m_i^2\right)^2.$$

The quantity x_p, however, is different from zero in the BM solution, although it vanishes at the lower band edge in the quenched case where it coincides with the Parisi solution [14, 27]. The inclusion of the projection term modifies the eigenvalues of B; however, the $N - 1$ higher eigenvalues remain confined in the original band, i.e. they remain positive, only the lowest eigenvalue is split out of the band of a finite amount possibly becoming negative. Therefore, the TAP solutions can only be minima or saddles of order one depending on the sign of the isolated eigenvalue or equivalently of the factor $(1 + \langle\alpha|B^{-1}|\alpha\rangle)$; if this term is zero the isolated eigenvalue vanishes and the solution is an inflection point. Starting from equation (6) applied to $B = A - P$ it is easy to prove that the factor controlling the determinant of the Hessian can also be written as

$$1 + \langle\alpha|B^{-1}|\alpha\rangle = \frac{1}{1 - \langle\alpha|A^{-1}|\alpha\rangle}. \tag{7}$$

Thus we can express the factor controlling the determinant of the Hessian A in terms of the susceptibility matrix $X = A^{-1}$, and the key object controlling the sign of the isolated eigenvalue is the quantity L defined as

$$L = \frac{1}{N} \sum_{ij} m_i X_{ij} m_j. \tag{8}$$

In [22] it is proven that this quantity is divergent on the BM solution because of the SUSY violation, therefore the isolated eigenvalue is zero. On the numerical ground the situation is not completely clear [28, 29] although some evidence in favour of the BM solution and of its predictions has recently been exhibited in [29]. Furthermore, an argument can be advanced for the continuity of the TAP complexity at $T = 0$ [31], a property that is satisfied by the BM solution.

The study of the isolated eigenvalue, however, has not solved all the problems connected to the BM solutions. For instance, it is not clear why the whole curve $\Sigma(f)$ is dominated by solutions with a zero eigenvalue and not only the point where it is maximal [22]. It is also known [20] that the prefactor of the exponential contribution to the total complexity vanishes at all orders in power of $1/N$, a result that was extended to the whole curve $\Sigma(f)$ in [22]. As discussed in [20, 22] this may be a good thing in order to recover the Morse theorem prediction but leaves open the question of whether we can really identify the complexity with the exponential contribution if its prefactor is zero.

Most importantly the presence of a zero eigenvalue complicates the behaviour of the TAP solutions upon changing the external parameters, and in particular the possibility of continuing them or not, a property that is crucial in setting up the Cavity method in order to apply the theory to a wider class of problems. In the following we will answer these questions, and as a byproduct we will eventually obtain the physical meaning of the parameters λ and Δ of the BM theory that have remained obscure until now.

We will show that in order to understand all the features of the BM solution for the TAP complexity we must consider two-order parameters, the free energy f and the self-overlap q and not only f. Thus we must study the function $\Sigma(f, q)$ or equivalently the function $\tilde{\Sigma}(u, \lambda_q)$ defined as

$$\Sigma(f, q) = \ln \sum_{\alpha} \delta(q - q_{\alpha\alpha})\delta(f - f_\alpha); \qquad \tilde{\Sigma}(u, \lambda_q) = \ln \sum_{\alpha} \mathrm{e}^{-u f_\alpha - \lambda_q q_{\alpha\alpha}}. \tag{9}$$

The extremization with respect to q and f, in order to compute the total complexity, requires to set $u = 0$ and $\lambda_q = 0$ in the computation of $\tilde{\Sigma}$, however we shall show that *we cannot set* $\lambda_q = 0$ *from the beginning but we must study the* $\lambda_q \neq 0$ *case and take the limit, indeed* λ_q *appears in products with quantities that are divergent in the limit* $\lambda_q \to 0$ *thus yielding a finite contribution. One of these diverging quantities is precisely the parameter L introduced above whose inverse controls the sign of the isolated eigenvalue.* In contrast there are no divergences associated with the parameter u that can be safely set to zero from the beginning. Furthermore the singular behaviour associated with the limit $\lambda_q \to 0$ is present for each u thus explaining why the whole line $\Sigma(f)$ of the BM solution is singular and the prefactor of the exponential is zero. In the following we shall consider only the case $u = 0$ and study the curve $\Sigma(q)$ or its Legendre transform $\tilde{\Sigma}(\lambda_q)$. We recall that in the thermodynamic limit the two formulations are completely equivalent because the function $\tilde{\Sigma}(\lambda_q)$ defined above is dominated by the solution at a given value of q that fixes an unequivocal correspondence between λ_q and q , the two functions are indeed Legendre transforms one of the other:

$$\lambda_q = \frac{\mathrm{d}\Sigma}{\mathrm{d}q}; \qquad q = -\frac{\mathrm{d}\tilde{\Sigma}}{\mathrm{d}\lambda_q}; \tag{10}$$

$$\tilde{\Sigma} = \Sigma - \lambda_q q. \tag{11}$$

We shall find that the isolated eigenvalue is proportional to λ_q, thus the determinant of the Hessian of the TAP solution is zero at $\lambda_q = 0$ as already shown in [21, 22] *but it is finite for* $\lambda_q \neq 0$. Considering equation (10) we see that the point q^* where the curve $\Sigma(q)$ is maximal separates two regions: on one side we have the minima while on the other side we have the saddles of order one, the two regions touch at $q = q^*$ that corresponds to solutions that are inflection points in the TAP landscape.

As soon as $\lambda_q \neq 0$ the corresponding TAP solutions have a non-vanishing determinant of the Hessian and thus it is possible to continue them upon changes of the external fields, e.g. temperature and magnetic field. In particular we will be able to apply the cavity method by studying the continuation of each TAP solution when a new spin is added. The use of the cavity method in order to compute the complexity is discussed in the literature, see, e.g., [7, 15], the crucial modification presented in the following section is that to recover the BM prediction for the TAP complexity we need to weigh the states with the overlap rather than with the free energy.

In section 3 we will discuss the $\lambda_q \neq 0$ case within the SUSY context, this will enable us to clarify many points, from the behavior of the prefactor to the meaning of the SUSY Ward identities. We will find that the BM solution does satisfy the SUSY Ward identities at each λ_q; however, because of the aforementioned divergences in the limit $\lambda_q \to 0$ these Ward identities are different from those obtained by simply $\lambda_q = 0$ (e.g. the Ward identities considered up to now in the literature see. e.g., [12, 14, 20]), in other words *if we carefully take the limit* $\lambda_q \to 0$ *we find that the non-SUSY solution satisfies the correct SUSY identities*.

Before entering the computation of the TAP complexity within the cavity method, we recall what are the modifications that must be made to study the $\lambda_q \neq 0$ case in the standard BM computation of [4]. To compute the function $\tilde{\Sigma}(\lambda_q)$ we must simply add to expression (15) of [4] (i.e. expression (5) above) a term equal to $\lambda_q q$ and then extremize at fixed λ_q with respect to the parameters q, B, Δ and λ (not to be confused with λ_q). The resulting equation for B, Δ and λ is the same as in the $\lambda_q = 0$ case while the equation obtained by extremizing with respect to q is

$$\lambda_q = -\lambda + \Delta + B - \frac{1}{2q}\left(1 - \frac{\langle(\tanh^{-1} m_0 - \Delta m_0)^2\rangle}{q\beta^2}\right). \tag{12}$$

Consistently upon computing the complexity one finds that λ_q and q satisfy relations (10).

2. Cavity method for the TAP complexity

2.1. The continuation of a TAP solution when a new spin is added

The TAP equations of the system with N spins read as

$$-\beta \sum_{i \neq j} m_j + \frac{\beta^2}{N} \sum_j \left(1 - m_j^2\right) m_i + \tanh^{-1} m_i = 0. \tag{13}$$

We add a new spin m_0 to the system and make the following definitions:

$$m_i^{(N+1)} = m_i + \delta m_i \tag{14}$$

$$\Delta q = Q^{(N+1)} - Q^{(N)} = m_0^2 + 2\sum_i m_i \delta m_i + \sum \delta m_i^2 \tag{15}$$

$$R(\delta m_i) = \tanh^{-1} m_i^{(N+1)} - \tanh^{-1} m_i - \frac{1}{1 - m_i^2}\delta m_i = \frac{1}{2}\left(\frac{d}{dm_i}\frac{1}{1 - m_i^2}\right)\delta m_i^2 + O\left(\frac{1}{N^{3/2}}\right) \tag{16}$$

$$X_{ij}^{-1} = \left(\frac{1}{\beta}\frac{1}{1 - m_i^2} + \frac{\beta}{N}\sum_{j=1}\left(1 - m_j^2\right)\right)\delta_{ij} - J_{ij} - \frac{2\beta}{N} m_i m_j. \tag{17}$$

We also define two fields and three parameters for the later use

$$H_1 = \sum_j J_{j0} m_j \tag{18}$$

$$H_2 = \sum_j J_{j0}\left(\sum_i X_{ij} m_i\right) \tag{19}$$

$$L = \frac{1}{N}\sum_{ij} m_i X_{ij} m_j \tag{20}$$

$$Z_1 = \frac{1}{N}\sum_{ij} m_i (X^2)_{ij} m_j \tag{21}$$

$$Z_2 = \frac{1}{N}\sum_{ij} (X_{ij} m_i)\left(\frac{d}{dm_i}\frac{1}{1 - m_i^2}\right)\sum_s X_{sj}^2 \tag{22}$$

$$X_{SG} = \frac{1}{N}\sum_j (X^2)_{jj}. \tag{23}$$

We adopt the convention that all the objects without the label $(N+1)$ are computed on the N system. From the TAP equations in the presence of the spin m_0 we can derive an *exact* expression for δm_i in implicit form:

$$\delta m_i = \sum_i X_{ij}\left(J_{j0}m_0 - \frac{\beta}{N}\big(1-m_0^2\big)m_j + \frac{\beta}{N}\sum_{s=1}\delta m_s^2 m_j - \frac{\beta}{N}(1-\Delta q)\delta m_j - \frac{1}{\beta}R(\delta m_j)\right). \tag{24}$$

From the previous expression we have

$$\delta m_i = \sum_i X_{ij}J_{j0}m_0 + O\left(\frac{1}{N}\right). \tag{25}$$

Therefore

$$\frac{1}{N}\sum_{s=1}\delta m_s^2 = \frac{1}{N}\frac{1}{N}\mathrm{Tr}X^2 m_0^2 = m_0^2\frac{1}{N}X_{SG} + \cdots. \tag{26}$$

Using this relation we can easily obtain from the exact expression (24) an expression for δm_i in explicit form valid to order $1/N$:

$$\delta m_i = \sum_j X_{ij}\left(J_{j0}m_0 - \frac{\beta}{N}\big(1-m_0^2\big)m_j + m_0^2\frac{\beta}{N}X_{SG}m_j \right.$$
$$\left. - \frac{m_0^2}{2\beta}\left(\frac{\mathrm{d}}{\mathrm{d}m_j}\frac{1}{1-m_j^2}\right)\left(\sum_s X_{sj}J_{s0}\right)^2\right) + O\left(\frac{1}{N^{3/2}}\right). \tag{27}$$

2.2. *The expression for m_0 at order $O(1)$ and the susceptibilities*

The equation for m_0 at order $O(1)$ is

$$\tanh^{-1} m_0 + \beta^2(1-q)m_0 - \beta\sum_i J_{i0}m_i^{(N+1)} = 0 \tag{28}$$

the field $H_1^{(N+1)}$ can be expressed in terms of the field H_1

$$H_1^{(N+1)} = H_1 + \sum_i J_{i0}\delta m_i = H_1 + \sum_{ij} J_{i0}X_{ij}J_{j0}m_0 + O\left(\frac{1}{N}\right) = H_1 + m_0\frac{1}{N}\,\mathrm{Tr}X + O\left(\frac{1}{N}\right). \tag{29}$$

Introducing the variable B

$$B = \beta^2(1-q) - \frac{\beta}{N}\,\mathrm{Tr}X, \tag{30}$$

we can express m_0 in terms of the field H_1 computed on the system with N spins

$$\tanh^{-1} m_0 + Bm_0 - \beta H_1 = 0. \tag{31}$$

In the presence of a magnetic field acting on the spin m_0 we have

$$\tanh^{-1} m_0 + Bm_0 - \beta H_1 - \beta h_0 = 0, \tag{32}$$

and we can obtain the following susceptibilities at order $O(1)$:

$$\frac{\mathrm{d}m_0}{\mathrm{d}h_0} = X_{00} = \beta\left(\frac{1}{1-m_0^2} + B\right)^{-1} + \cdots \tag{33}$$

$$\frac{\mathrm{d}^2 m_0}{\mathrm{d}h_0^2} = \frac{\mathrm{d}}{\mathrm{d}m_0}\left(\frac{\mathrm{d}h_0}{\mathrm{d}m_0}\right)^{-1}\frac{\mathrm{d}m_0}{\mathrm{d}h_0} = -\left(\frac{\mathrm{d}}{\mathrm{d}m_0}\frac{1}{1-m_0^2}\right)\beta\left(\frac{1}{1-m_0^2} + B\right)^{-2}\frac{\mathrm{d}m_0}{\mathrm{d}h_0} + \cdots. \tag{34}$$

The mixed susceptibility can be obtained from the explicit expression of δm_i valid at order $O(1/N)$, equation (27):

$$X_{i0}^{(N+1)} = \frac{\mathrm{d}m_i^{(N+1)}}{\mathrm{d}h_0} \tag{35}$$

$$= \sum_j X_{ij}\left(J_{j0}X_{00} + \frac{2\beta}{N}m_0 X_{00} m_j + 2m_0 X_{00}\frac{\beta}{N}X_{SG}m_j\right.$$

$$\left. - \frac{m_0 X_{00}}{\beta}\left(\frac{\mathrm{d}}{\mathrm{d}m_j}\frac{1}{1-m_j^2}\right)\left(\sum_s X_{sj}J_{s0}\right)^2\right) + O\left(\frac{1}{N^{3/2}}\right). \tag{36}$$

2.3. *The self-overlap shift*

The shift of the self-overlap reads as

$$\Delta q = Q^{(N+1)} - Q^{(N)} = m_0^2 + 2\sum_i m_i \delta m_i + \sum \delta m_i^2. \tag{37}$$

The term $\sum_i m_i \delta m_i$ can be evaluated at order $O(1)$ starting from the expression for δm_i at order $O(1/N)$, equation (27):

$$\sum_i m_i \delta m_i \tag{38}$$

$$= \sum_{ij} m_i X_{ij}\left(J_{j0}m_0 - \frac{\beta}{N}\left(1-m_0^2\right)m_j + m_0^2\frac{\beta}{N}X_{SG}m_j\right.$$

$$\left. - \frac{m_0^2}{2\beta}\left(\frac{\mathrm{d}}{\mathrm{d}m_j}\frac{1}{1-m_j^2}\right)\left(\sum_s X_{sj}J_{s0}\right)^2\right) + O\left(\frac{1}{N^{1/2}}\right) \tag{39}$$

$$= m_0 H_2 - \beta\left(1-m_0^2\right)L + m_0^2\beta X_{SG}L - \frac{m_0^2}{2\beta}Z_2 + O\left(\frac{1}{N^{1/2}}\right). \tag{40}$$

The fourth term has been simplified in the following way:

$$\sum_{ij} m_i X_{ij}\left(\frac{\mathrm{d}}{\mathrm{d}m_j}\frac{1}{1-m_j^2}\right)\left(\sum_s X_{sj}J_{s0}\right)^2$$

$$= \frac{1}{N}\sum_{ij} m_i X_{ij}\left(\frac{\mathrm{d}}{\mathrm{d}m_j}\frac{1}{1-m_j^2}\right)\sum_s X_{js}^2 + \cdots = Z_2 + \cdots. \tag{41}$$

The expression for $\sum_i \delta m_i^2$ was derived above, see equation (26). Collecting all the terms we obtain the following expression for Δq valid at order $O(1)$:

$$\Delta q = m_0^2 + 2m_0 H_2 - 2\beta\left(1-m_0^2\right)L + 2m_0^2\beta X_{SG}L - m_0^2\frac{1}{\beta}Z_2 + m_0^2 X_{SG} + O(1/N^{1/2}) \tag{42}$$

2.4. The distribution of the fields

The two fields entering the computation are

$$H_1 = \sum_j J_{j0} m_j \tag{43}$$

$$H_2 = \sum_j J_{j0} \left(\sum_i X_{ij} m_i \right). \tag{44}$$

Before the reweighting they have a Gaussian distribution with covariances

$$\langle H_1^2 \rangle_{(N)} = \frac{1}{N} \sum_j m_j^2 = q \tag{45}$$

$$\langle H_2^2 \rangle_{(N)} = \frac{1}{N} \sum_j \left(\sum_i X_{ij} m_i \right)^2 = \frac{1}{N} \sum_{ij} m_i (X^2)_{ij} m_j = Z_1 \tag{46}$$

$$\langle H_1 H_2 \rangle_{(N)} = \frac{1}{N} \sum_j m_j \left(\sum_i X_{ij} m_i \right) = L. \tag{47}$$

Within the cavity method (see [1], chapter V) the average over the fields must be defined weighting the states properly

$$P(H_1, H_2) = \frac{\sum_\alpha \delta\big(H_1 - H_1^\alpha\big) \delta\big(H_2 - H_2^\alpha\big) \, \mathrm{e}^{-u f_\alpha - \lambda_q q_{\alpha\alpha}}}{\sum_\alpha \mathrm{e}^{-u f_\alpha - \lambda_q q_{\alpha\alpha}}}. \tag{48}$$

When a new spin is added to the system the weight of each state changes because of the variation of its free energy and self-overlap. The distribution of the fields computed above corresponds to weight the states with the old weights corresponding to the system with N spins, in order to obtain the correct distribution we must reweight it considering the weight change due to the shift in free energy and overlap of each state which depend only on the fields H_1 and H_2. Thus after the reweighting the distribution of the fields is proportional to

$$P^{(N+1)}(H_1, H_2) \propto \exp\left[-\frac{1}{2} (H_1 H_2) C^{-1} \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} - \lambda_q \Delta q \right] \tag{49}$$

where according to equations (45)–(47)

$$C = \begin{pmatrix} q & L \\ L & Z_1 \end{pmatrix}. \tag{50}$$

At the end of the computation we expect to find that λ_q is proportional to the derivative of the complexity with respect to q.

$$\lambda_q = \frac{\mathrm{d}\Sigma}{\mathrm{d}q}. \tag{51}$$

The variation of the self-overlap Δq was computed above, see equation (42). Neglecting the irrelevant constant term it can be written as

$$\Delta q = \frac{a}{2} m_0^2 + 2 m_0 H_2 \quad \text{with} \quad \frac{a}{2} = 1 + 2\beta L + 2\beta X_{SG} L + X_{SG} - \frac{1}{\beta} Z_2. \tag{52}$$

The magnetization is univoquely determined by H_1 according to the previously derived relation:

$$\tanh^{-1} m_0 + B m_0 - \beta H_1 = 0. \tag{53}$$

Performing the integration over H_2 and changing variable from H_1 to m_0 we obtain that the reweighted distribution of H_1 is proportional to

$$\left(\frac{1}{1-m_0^2}+B\right)\exp\left[-\frac{(\tanh^{-1}m_0+Bm_0+2\beta L\lambda_q m_0)^2}{2q\beta^2}-\lambda_q\frac{a}{2}m_0^2+2Z_1\lambda_q^2m_0^2\right]\mathrm{d}m_0. \quad (54)$$

We introduce the new variables

$$\Delta = -B - 2\beta\lambda_q L \quad (55)$$

$$\lambda = -\lambda_q\frac{a}{2} + 2Z_1\lambda_q^2. \quad (56)$$

Note that if $B = 0L$ and λ_q are inversely proportional meaning that if Δ remains finite when λ_q goes to zero (which is the case for the BM solution and not for the SUSY solution) then L must diverge accordingly meaning the isolated eigenvalue is vanishing. With the above change of variables the reweighted distribution takes the BM form

$$P^{(N+1)}(m_0) = K\left(\frac{1}{1-m_0^2}+B\right)\exp\left[-\frac{(\tanh^{-1}m_0-\Delta m_0)^2}{2q\beta^2}+\lambda m_0^2\right]\mathrm{d}m_0 \quad (57)$$

where K is the normalization constant.

A word of caution is in order here. The fluctuations of the field H_1 and H_2 considered in equations (45)–(47) and described by the distribution function (49) are not fluctuations over the disorder but over the different TAP solutions at a given realization of the disorder. In general this leads to the fact that the fields before the reweighting are Gaussian but with non-zero means H_{1J} and H_{2J} depending on the disorder, see equation (V.25) on page 71 in [1]. Then the average over the disorder is carried on considering the distribution function of H_{1J} and H_{2J} that are again Gaussian with zero mean but with a covariance matrix that is in general different from zero. In the standard case, see equation (V.37) in [1], the variance is proportional to the overlap between different states and analogous relationship can be obtained in our case. However, if we assume that different TAP solutions of the same sample are uncorrelated, the disorder variances of H_{1J} and H_{2J} become zero, that is we have precisely $H_{1J} = 0$ and $H_{2J} = 0$ on each sample. This corresponds in the replica language to making a 1RSB ansatz with $q_0 = 0$. If H_{1J} and H_{2J} are zero on each sample, equation (49) can be identified with the distribution over the TAP solutions and over the disorder. Indeed in the following we make this identification. This is correct as long as we want to reproduce the result of the annealed computation of the TAP complexity. Instead if we keep non-zero H_{1J} and H_{2J} we can reproduce the so-called replica-symmetric quenched computation of the complexity of BM, see equations (19)–(21) in [4]. We recall however that, as shown in [4], the annealed computation is correct for what concerns the total complexity and one can hope that this simplification holds in other models as well.

2.5. The self-consistency equations

The self-consistency equation for the various parameters of the theory are

$$q = \langle m_0^2\rangle \quad (58)$$

$$L = \left\langle m_0\sum_i X_{i0}^{(N+1)}m_i^{(N+1)}\right\rangle \quad (59)$$

$$Z_1 = \left\langle\left(\sum_i X_{i0}^{(N+1)}m_i^{(N+1)}\right)^2\right\rangle \quad (60)$$

$$Z_2 = \left\langle \sum_i X_{i0}^{(N+1)} m_i^{(N+1)} \left(\frac{d}{dm_0} \frac{1}{1-m_0^2} \right) \left(\sum_{j=0} \left(X_{j0}^{(N+1)} \right)^2 \right) \right\rangle \tag{61}$$

$$\frac{1}{N} \mathrm{Tr} X = \langle X_{00} \rangle \tag{62}$$

$$X_{SG} = \left\langle X_{00}^2 \right\rangle + \left\langle \sum_{j=1} \left(X_{j0}^{(N+1)} \right)^2 \right\rangle. \tag{63}$$

The first equation is simply the equation for q in the BM theory. The self-consistency equation for $\mathrm{Tr}X/N$ gives the BM equation for B, indeed recalling the definition of B, equation (30), and the expression of X_{00}, equation (33), we obtain

$$B = \beta^2(1-q) - \beta\langle X_{00}\rangle = \beta^2 \left(1 - q - \left\langle \left(\frac{1}{1-m_0^2} + B \right)^{-1} \right\rangle \right). \tag{64}$$

Recalling the expression for $X_{0i}^{(N+1)}$, equation (36) we can evaluate the equation for X_{SG}

$$X_{SG} = \left\langle X_{00}^2 \right\rangle + \left\langle \sum_{j=1} \left(X_{j0}^{(N+1)} \right)^2 \right\rangle \tag{65}$$

$$\begin{aligned} X_{SG} &= \left\langle \beta^2 \left(\frac{1}{1-m_0^2} + B \right)^{-2} \right\rangle + \left\langle \beta^2 \left(\frac{1}{1-m_0^2} + B \right)^{-2} \sum_{j=1} \left(\sum_i J_{i0} X_{ji} + O(1/N) \right)^2 \right\rangle \\ &= \left\langle \beta^2 \left(\frac{1}{1-m_0^2} + B \right)^{-2} \right\rangle (1 + X_{SG}). \end{aligned} \tag{66}$$

Therefore

$$X_{SG} = \frac{\left\langle \beta^2 \left(\frac{1}{1-m_0^2} + B \right)^{-2} \right\rangle}{1 - \left\langle \beta^2 \left(\frac{1}{1-m_0^2} + B \right)^{-2} \right\rangle}. \tag{67}$$

In the case $B = 0$ the expression simplifies to

$$X_{SG} = \frac{1}{x_p} - 1 \tag{68}$$

with

$$x_p = 1 - \left\langle \beta^2 \left(1 - m_0^2 \right)^2 \right\rangle. \tag{69}$$

As in [1] we can compute the object

$$X_{SG} - \left\langle X_{00}^2 \right\rangle = \frac{(1-x_p)^2}{x_p}. \tag{70}$$

In this way we recover the stability condition $x_p \geqslant 0$ of the BM theory in the $B = 0$ case as a positivity condition over $X_{SG} - \left\langle X_{00}^2 \right\rangle$.

In order to compute the self-consistency equations for the parameter L, Z_1 and Z_2 we need to determine the quantity $\sum_i X_{i0}^{(N+1)} m_i^{(N+1)}$ at order $O(1)$. This can be done starting from the expression of δm_1 and $X_{0i}^{(N+1)}$ derived above, that is equations (27) and (36). We do not report the result which turns out to be equal to half the derivative of Δq with respect to h_0 as it should since $X_{i0}^{(N+1)} = \mathrm{d}m_i^{(N+1)}/\mathrm{d}h_0$. In order to proceed in the computation it is also useful to note that according to equation (32) derivatives with respect to h_0 are equivalent to derivatives with respect to H_1; therefore we have

$$\sum_i X_{i0}^{(N+1)} m_i^{(N+1)} = \frac{1}{2}\frac{\mathrm{d}\Delta q}{\mathrm{d}h_0} = \frac{1}{2}\frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}. \tag{71}$$

With this equation we can perform integration by parts and do not need the explicit expression of $\sum_i X_{i0}^{(N+1)} m_i^{(N+1)}$. We rewrite the self-consistency equations for L, Z_1 and Z_2 as

$$L = \frac{1}{2}\left\langle m_0 \frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}\right\rangle \tag{72}$$

$$Z_1 = \frac{1}{4}\left\langle \left(\frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}\right)^2\right\rangle \tag{73}$$

$$Z_2 = \left\langle \frac{1}{2}\frac{\mathrm{d}\Delta q}{\mathrm{d}h_0}\left(\frac{\mathrm{d}}{\mathrm{d}m_0}\frac{1}{1-m_0^2}\right)\beta^2\left(\frac{1}{1-m_0^2}+B\right)^{-2}\right\rangle (1+X_{SG}). \tag{74}$$

In the last expression we have used the expression for $\sum_{i=0}\left(X_{0i}^{(N+1)}\right)^2$ previously derived in the computation of X_{SG}, see equation (66).

2.6. The L equation

The self-consistency equation for the parameter L reads as

$$L = \frac{1}{2}\left\langle m_0 \frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}\right\rangle. \tag{75}$$

To compute this expression we shall use integration by parts. In the following we will use the shorthand expression for the Gaussian part of the distribution of the fields:

$$G[H_1, H_2] = \exp\left[-\frac{1}{2}(H_1 \quad H_2)C^{-1}\begin{pmatrix}H_1\\ H_2\end{pmatrix}\right] \tag{76}$$

where C is defined in equation (50). This is essentially the distribution of H_1 and H_2 before the reweighting. We also recall that K is the normalization constant of the reweighted distribution. Integrating by parts we have

$$L = -K\int m_0 \frac{1}{2\lambda_q} G[H_1, H_2]\frac{\mathrm{d}}{\mathrm{d}H_1}\exp[-\lambda_q \Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2 \tag{77}$$

$$= \frac{1}{2\lambda_q}\left(\langle X_{00}\rangle + K\int m_0\left(\frac{\mathrm{d}}{\mathrm{d}H_1}G[H_1, H_2]\right)\exp[-\lambda_q\Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2\right). \tag{78}$$

Taking the derivative of $\mathrm{d}G[H_1, H_2]/\mathrm{d}H_1$ and making the integration over H_2 we obtain

$$2\beta\lambda_q L = \beta^2(1-q) - B - \left\langle m_0 \frac{\tanh^{-1} m_0 + Bm_0 + 2\beta L\lambda_q m_0}{q}\right\rangle. \tag{79}$$

Recalling the change of variables

$$\Delta = -B - 2\beta\lambda_q L \tag{80}$$

we obtain

$$\Delta = -\frac{\beta^2}{2}(1-q) + \frac{1}{2q}\langle m_0 \tanh^{-1} m_0\rangle. \tag{81}$$

Thus the self-consistency equation for L gives an equation identical to that obtained in the BM theory extremizing with respect to Δ, see [4].

2.7. The Z_1 equation

The self-consistency equation for the parameter Z_1 reads as

$$Z_1 = \frac{1}{4}\left\langle \left(\frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}\right)^2 \right\rangle. \tag{82}$$

Integrating by parts we have

$$\begin{aligned} Z &= \frac{K}{4}\int \left(\frac{\mathrm{d}\Delta q}{\mathrm{d}H_1}\right)^2 G[H_1, H_2]\exp[-\lambda_q \Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2 \\ &= \frac{K}{4\lambda_q^2}\int \left(\frac{\mathrm{d}^2}{\mathrm{d}H_1^2}G[H_1, H_2]\right)\exp[-\lambda_q \Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2 \\ &\quad + \frac{K}{4\lambda_q}\int \frac{\mathrm{d}^2\Delta q}{\mathrm{d}H_1^2}G[H_1, H_2]\exp[-\lambda_q \Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2. \end{aligned} \tag{83}$$

A simple computation shows that the first term gives the following contribution:

$$\begin{aligned} &\frac{K}{4\lambda_q^2}\int \left(\frac{\mathrm{d}^2}{\mathrm{d}H_1^2}G[H_1, H_2]\right)\exp[-\lambda_q \Delta q]\,\mathrm{d}H_1\,\mathrm{d}H_2 \\ &\quad = \frac{1}{4\lambda_q^2}\left(-\frac{1}{q}\left(1 - \frac{\langle(\tanh^{-1} m_0 - \Delta m_0)^2\rangle}{q\beta^2}\right)\right). \end{aligned} \tag{84}$$

The second term in (83) requires more care. We start recalling the expression for Δq, equation (42), neglecting as usual the constant term. As in section 2.4 we write it as

$$\Delta q = \frac{a}{2}m_0^2 + 2m_0 H_2 \quad \text{with} \quad \frac{a}{2} = 1 + 2\beta L + 2\beta X_{SG} L + X_{SG} - \frac{1}{\beta}Z_2. \tag{85}$$

Instead of considering the derivatives with respect to H_1 we consider the derivatives with respect to h_0 as originally done in equation (71)

$$\begin{aligned} \frac{\mathrm{d}^2\Delta q}{\mathrm{d}H_1^2} &= \frac{\mathrm{d}^2\Delta q}{\mathrm{d}h_0^2} = \frac{\mathrm{d}}{\mathrm{d}h_0}\left(\frac{\mathrm{d}\Delta q}{\mathrm{d}m_0}\frac{\mathrm{d}m_0}{\mathrm{d}h_0}\right) = \frac{\mathrm{d}}{\mathrm{d}h_0}\left((am_0 + 2H_2)\frac{\mathrm{d}m_0}{\mathrm{d}h_0}\right) \\ &= a\left(\frac{\mathrm{d}m_0}{\mathrm{d}h_0}\right)^2 + (2m_0 + 2H_2)\frac{\mathrm{d}^2 m_0}{\mathrm{d}h_0^2}. \end{aligned} \tag{86}$$

The second term can be simplified using equation (34)

$$(2m_0 + 2H_2)\frac{\mathrm{d}^2 m_0}{\mathrm{d}h_0^2} = -\left(\frac{\mathrm{d}}{\mathrm{d}m_0}\frac{1}{1-m_0^2}\right)\beta\left(\frac{1}{1-m_0^2} + B\right)^{-2}(2m_0 + 2H_2)\frac{\mathrm{d}m_0}{\mathrm{d}h_0} \tag{87}$$

$$= -\left(\frac{\mathrm{d}}{\mathrm{d}m_0}\frac{1}{1-m_0^2}\right)\beta\left(\frac{1}{1-m_0^2} + B\right)^{-2}\frac{\mathrm{d}\Delta q}{\mathrm{d}h_0}. \tag{88}$$

Recalling the self-consistency equation for Z_2, equation (74), we have that the average of the second term in (86) is simply given by

$$\left\langle (2m_0 + 2H_2)\frac{\mathrm{d}^2 m_0}{\mathrm{d}h_0^2}\right\rangle = -\frac{2Z_2}{\beta}\frac{1}{1 + X_{SG}}. \tag{89}$$

To evaluate the average of the first term in (86) we must recall the expression of the susceptibility, equation (33), and the expression of X_{SG}, equation (67), we have

$$\left\langle \left(\frac{\mathrm{d}m_0}{\mathrm{d}h_0}\right)^2 \right\rangle = \frac{X_{SG}}{1 + X_{SG}}. \tag{90}$$

Summing up the two terms and recalling the expression for a, equation (85), we obtain

$$\left\langle \frac{\mathrm{d}^2\Delta q}{\mathrm{d}h_0^2} \right\rangle = a\frac{X_{SG}}{1+X_{SG}} - \frac{2Z_2}{\beta}\frac{1}{1+X_{SG}} = a - 2 - 4\beta L. \tag{91}$$

Thus equation (83) reads as

$$Z_1 = \frac{1}{4\lambda_q^2}\left(-\frac{1}{q}\left(1 - \frac{\langle(\tanh^{-1} m_0 - \Delta m_0)^2\rangle}{q\beta^2}\right)\right) + \frac{1}{4\lambda_q}(a - 2 - 4\beta L). \tag{92}$$

Multiplying both sides by $2\lambda_q^2$ we obtain

$$2\lambda_q^2 Z_1 = -\frac{1}{2q}\left(1 - \frac{\langle(\tanh^{-1} m_0 - \Delta m_0)^2\rangle}{q\beta^2}\right) + \lambda_q\left(\frac{a}{2} - 1 - 2\beta L\right). \tag{93}$$

Now recalling the definition of the variables λ and Δ introduced in section 2.4

$$\Delta = -B - 2\beta\lambda_q L \tag{94}$$

$$\lambda = -\lambda_q\frac{a}{2} + 2Z_1\lambda_q^2 \tag{95}$$

we obtain the equation

$$\lambda_q = -\lambda + \Delta + B - \frac{1}{2q}\left(1 - \frac{\langle(\tanh^{-1} m_0 - \Delta m_0)^2\rangle}{q\beta^2}\right). \tag{96}$$

This is precisely the equation corresponding to the extremization of the BM action at a fixed value of the self-overlap q reported in the introduction and we have the correct identification between λ_q and the derivative of $\Sigma(q)$ with respect to q:

$$\lambda_q = \frac{\mathrm{d}\Sigma}{\mathrm{d}q}. \tag{97}$$

We do not report the explicit expression of the complexity that can be obtained taking into account the rescaling of the temperature associated with the process of adding a spin without changing the couplings [1].

3. On supersymmetry

By constraining the solutions to have a given overlap q or equivalently by weighting the sum over solutions with a weight proportional to $\mathrm{e}^{-\lambda_q q}$ we break the supersymmetry of the problem. Considering the bell-shaped curve described by $\Sigma(q)$ we see that at the maximum the action is SUSY while for q different from the maximum value q^* or equivalently for $\lambda_q \neq 0$ the action is not SUSY but the SUSY violation takes a very simple form.

In the following we will see that at any value of λ_q it is possible to write some SUSY Ward identities that reduce to the SUSY relations in the limit $\lambda_q \to 0$. In these relations the parameter λ_q multiplies some quantities that diverge in the limit $\lambda_q \to 0$. Therefore taking carefully this limit we obtain relations that are different from those obtained simply putting $\lambda_q = 0$ in the equations.

The point $\Sigma(q^*)$ corresponding to the maximum of $\Sigma(q)$ is singular, there is a zero eigenvalue that is definitively different from zero a finite λ_q and changes sign at $\lambda_q = 0$. This can be clearly seen because this eigenvalue is proportional to the inverse of the physical parameter L that diverges as $\lambda_q \to 0$ since $L \propto \Delta/\lambda_q$. Therefore for each $\lambda_q \neq 0$ the determinant of the TAP solutions is strictly different from zero and we can apply the cavity method by considering the continuation of each solution when a new spin is added. Note that the sign of the determinant of the solutions is unequivocally defined on the left and on the right of q^* therefore we can safely remove the modulus of the determinant from the standard computation of the complexity and the integral of the macroscopic action will also change sign at q^*.

At finite N the only points where solutions can rise or die are on the line $\lambda_q = 0$, then adding or removing spins the solutions move on the curve $\Sigma(q, f)$ since their self-overlap q and free energy f change with N thus leading to the correct behavior of the solutions as $\mathrm{e}^{N\Sigma(q,f)}$.

Since the SUSY is broken for $\lambda_q \neq 0$ the determinant of the fermionic fluctuation is no longer zero, much as the isolated eigenvalue. Thus at each $\lambda_q \neq 0$ we have a finite prefactor to the exponential contribution, whose expansion in power of $1/N$ vanishes at all orders in the limit $\lambda_q \to 0$. If one does not consider the $\lambda_q \to 0$ limit the vanishing of the prefactor at $\lambda_q = 0$ is a problem since it is not clear if this changes the behaviour of the complexity. One should be able to prove that even if its expansion vanishes at all orders the prefactor is definitively different from zero at each finite N. Note that this can happen for instance if the prefactor takes the form e^{-aN} thus changing the exponential contribution. Instead the fact that the prefactor is always finite at $\lambda_q \neq 0$ means that the exponential contribution is dominant and allows us to bypass this problem, indeed the identification of the total complexity with the BM complexity can be safely obtained considering the limit $\lambda_q \to 0$ of $\tilde{\Sigma}(\lambda_q)$ or equivalently the limit $q \to q^*$ of $\Sigma(q)$.

At the macroscopic level the SUSY-like relationships we will obtain are not relationships between the macroscopic parameters, instead they are relationships that connect the macroscopic bosonic order parameters e.g. λ and Δ with averages of the fermionic macroscopic variables. In this way they yield independently of the cavity method the physical meaning of Δ and λ obtained in the previous section. Indeed the parameter L, Z_1 and Z_2 requires averages over the fermionic variables to be computed and the BRST relationship we will derive is equivalent to definitions (55) and (56).

3.1. Microscopic supersymmetry

If we consider the sum over solutions of the TAP equations weighting each solutions with a weigh proportional to $\mathrm{e}^{-\lambda_q q}$ we obtain that the result can be expressed (removing the modulus of the determinant) as an integral over the following action:

$$S = S(\{\overline{\psi_i}, \psi_i, m_i, x_i\}) - \lambda_q \sum_i m_i^2. \tag{98}$$

Here and in the following section we will adopt the notation of [22]. The first term is invariant under the microscopic BRST derivative D_{micro} induced by the following transformation [16–18]:

$$\delta m_i = \epsilon \psi_i \qquad \delta \bar{\psi}_i = -\epsilon x_i \tag{99}$$

$$D = \psi_i \partial_{m_i} - x_i \partial_{\bar{\psi}_i}. \tag{100}$$

The action is no longer invariant but the variation is easily computed from the previous relations

$$DS = -2\lambda_q \sum_i m_i \psi_i. \tag{101}$$

As a consequence the average of a BRST derivative is no longer zero but it is given by

$$\langle DO \rangle = \langle DSO \rangle = -2\lambda_q \left\langle \sum_i m_i \psi_i O \right\rangle. \tag{102}$$

Thus we can obtain Ward identities also at $\lambda_q \neq 0$; if we put $O = m_i\overline{\psi_j}$ we obtain the following relationship:

$$-\langle m_i x_j\rangle = \langle\overline{\psi_j}\psi_i\rangle - 2\lambda_q\left\langle m_i\sum_k\overline{\psi_j}\psi_k m_k\right\rangle. \tag{103}$$

If we set $\lambda_q = 0$ in the previous relation we obtain a standard SUSY relationship, instead it is crucial to consider the limit $\lambda_q \to 0$ because the term multiplied by λ_q diverges in this limit leading to a finite result.

Let us show that the previous relation follows naturally if we assume that the solutions can be continued. In this case we have that

$$\begin{aligned}\frac{\partial}{\partial h_j}\left(\sum_\alpha m_i^\alpha\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}\right) &= \sum_\alpha\frac{\partial m_i^\alpha}{\partial h_j}\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} - \sum_\alpha m_i^\alpha\lambda_q\frac{\partial q_{\alpha\alpha}}{\partial h_j}\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}\\ &= \sum_\alpha\frac{\partial m_i^\alpha}{\partial h_j}\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} - 2\lambda_q\sum_{\alpha k}m_i^\alpha X_{jk}^\alpha m_k^\alpha\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}.\end{aligned} \tag{104}$$

Recalling the following relationships we can easily see the equivalence between equations (103) (obtained considering the BRST relationship) and (104) (obtained assuming the possibility of continuing the TAP solutions)

$$\frac{\partial\langle m_i\rangle}{\partial h_j} = -\langle m_i x_j\rangle \tag{105}$$

$$\langle\overline{\psi_j}\psi_i\rangle = \sum_\alpha\frac{\partial m_i^\alpha}{\partial h_j}\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} \tag{106}$$

$$-2\lambda_q\langle m_i\sum_k\overline{\psi_j}\psi_k m_k\rangle = -2\lambda_q\sum_{\alpha k}m_i^\alpha X_{jk}^\alpha m_k^\alpha\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} \tag{107}$$

If we put $O = x_i\overline{\psi_j}$ in (102) we obtain the generalization of another well-known BRST relation to the case $\lambda_q \neq 0$

$$\langle x_i x_j\rangle = 2\lambda_q\langle x_i\sum_k\overline{\psi_j}\psi_k m_k\rangle. \tag{108}$$

On the other hand if we assume that the TAP solutions relevant for the weighted average $\sum_\alpha\exp[-\lambda_q q_{\alpha\alpha}]$ can be univoquely continued we have that

$$\frac{\partial^2}{\partial h_i\partial h_j}\sum_\alpha\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} = \frac{\partial}{\partial h_i}\sum_\alpha\left(-\lambda_q\frac{\partial q_{\alpha\alpha}}{\partial h_j}\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}\right) = \frac{\partial}{\partial h_i}\sum_\alpha\left(-2\lambda_q\sum_k X_{jk}^\alpha m_k^\alpha\,\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}\right). \tag{109}$$

The equivalence between equations (108) and (109) can be proven considering the following equations:

$$\frac{\partial^2}{\partial h_i\partial h_j}\sum_\alpha\mathrm{e}^{-\lambda_q q_{\alpha\alpha}} = \langle x_i x_j\rangle \tag{110}$$

$$\frac{\partial}{\partial h_i}\sum_\alpha\left(-2\lambda_q\sum_k X_{jk}^\alpha m_k^\alpha\mathrm{e}^{-\lambda_q q_{\alpha\alpha}}\right) = -2\lambda_q\frac{\partial}{\partial h_i}\left\langle\sum_k\overline{\psi_j}\psi_k m_k\right\rangle = 2\lambda_q\left\langle x_i\sum_k\overline{\psi_j}\psi_k m_k\right\rangle. \tag{111}$$

3.2. The SUSY relationships encode the physical meaning of the parameters

By summing equation (103) over $i = j$ we obtain

$$-\sum_i \langle m_i x_i \rangle - \sum_i \langle \overline{\psi_i} \psi_i \rangle = -2\lambda_q \left\langle \sum_{ik} m_i \overline{\psi_i} \psi_k m_k \right\rangle. \tag{112}$$

Recalling equation (107) we see that the rhs is nothing but the parameter $-2\lambda_q L$ where L was defined in (20). On the other hand the lhs of the previous equation can be expressed in terms of the macroscopic parameters Δ, B and λ, we have that it is equal to

$$-\sum_i \langle m_i x_i \rangle - \sum_i \langle \overline{\psi_i} \psi_i \rangle = \left\langle \frac{\Delta + B}{\beta} \right\rangle. \tag{113}$$

Thus equation (112) is equivalent to

$$\langle \Delta + B \rangle = -2\beta\lambda_q \langle L \rangle. \tag{114}$$

Therefore we see that the BRST relationships allow us to recover the physical meaning on the parameters Δ, B and λ of the standard TAP complexity computation without the cavity method. They relate these parameters to the physical parameters L, Z_1 and Z_2 and the relation is precisely that which is obtained within the cavity method. Indeed the last equation is equivalent to equation (55) that is the definition of Δ within the cavity method. Note once again the importance of taking the limit $\lambda_q \to 0$ instead of simply setting $\lambda_q = 0$. Since L is divergent in this limit the rhs of (114) remains finite. Instead if we put it to zero we obtain the standard SUSY solution $\Delta + B = 0$. In a similar way we can use the SUSY relation (108) to uncover the physical meaning of the parameter λ of the standard complexity computation in terms of the physical parameters L, Z_1, Z_2 and X_{SG}, obtaining a relation equivalent to equation (56) obtained within the cavity method.

3.3. Macroscopic supersymmetry

The results of the previous sections can be recovered also considering the macroscopic action obtained within the computation of the complexity. This depends on four bosonic and four fermionic parameters $\{r, t, \lambda, q, \overline{\rho}, \rho, \overline{\mu}, \mu\}$. In the presence of a forcing term $\lambda_q q$ the action is

$$S = S_0 - \lambda_q q. \tag{115}$$

The first term is invariant under a macroscopic BRST transformation given by

$$\begin{cases} \delta\mu = \frac{2\lambda}{\beta}\epsilon \\ \delta\rho = \left(\frac{r}{\beta} - \frac{t}{\beta}\right)\epsilon \\ \delta\overline{\mu} = 0 \\ \delta\overline{\rho} = 0 \\ \delta q = -\epsilon\frac{2\overline{\mu}}{\beta} \\ \delta\lambda = 0 \\ \delta r = -\epsilon(2\beta\overline{\mu} - \beta\overline{\rho}) \\ \delta t = \epsilon(-2\beta\overline{\mu} - \beta\overline{\rho}). \end{cases} \tag{116}$$

Therefore, the variation of the total action (115) is simply given by

$$\epsilon DS = -\lambda_q \delta q = \epsilon \frac{2\lambda_q}{\beta}\overline{\mu}. \tag{117}$$

Much as in the previous sections we have that the averages of a BRST derivative satisfy the following relation:

$$\langle DO\rangle = \langle DSO\rangle = -\frac{2\lambda_q}{\beta}\langle\overline{\mu}O\rangle. \tag{118}$$

Substituting for O and looking at the first two relations in (116) we have that

$$O = \mu \longrightarrow \left\langle\frac{2\lambda}{\beta}\right\rangle = -\frac{2\lambda_q}{\beta}\langle\overline{\mu}\mu\rangle \tag{119}$$

$$O = \rho \longrightarrow \left\langle\frac{r-t}{\beta}\right\rangle = -\frac{2\lambda_q}{\beta}\langle\overline{\mu}\rho\rangle. \tag{120}$$

As it was derived in [22] the rhs of the previous equation is proportional to the physical parameter L while skipping from the variable $\{r, t, \lambda, q\}$ to the BM variables $\{\Delta, B, \lambda, q\}$ the rhs is simply $(\Delta + B)/\beta$. Therefore, much as in the previous subsection, we obtain that the BRST relations within the macroscopic formalism encode the physical meaning of the parameters Δ and λ obtained within the cavity method.

4. Discussion

The appearance of an exponential number of TAP solutions below the critical temperature predicted by the Bray–Moore solution was firstly recognized by Kurchan [20] as a spontaneous breaking of the BRST supersymmetry. A natural way to study spontaneous symmetry breaking is to break explicitly the symmetry of the system through a small field and then to send the field to zero: the symmetry is spontaneously broken when this limit does not coincide with the symmetric result. This is precisely the approach we have followed here; indeed by ordering the TAP solutions with respect to the self-overlap we break explicitly the SUSY of the problem. The Morse theorem predicts that the sum over all TAP solutions weighted with the sign of their Hessian must sum up to one, i.e. its logarithm must be zero. Instead if we sum the solutions with a weight proportional to $e^{\lambda_q q}$ times the sign of their Hessian and then we send λ_q to zero we do not recover the symmetric (Morse theorem) result $\Sigma = 0$ but rather we get the BM complexity. The approach is not only useful to recover the BM theory within the cavity method but yields also some new physics: most importantly the fact that the BM theory describes also states with a non-vanishing isolated eigenvalue. More precisely, at each free energy the dominant states are marginal but there is also an exponential number of states with non-zero isolated eigenvalue.

Within this approach we have been able to eventually obtain the physical interpretation of the parameters Δ and λ of the BM theory in terms of physical objects like L, Z_1, Z_2 and X_{SG}. The extremization with respect to q corresponds to the case $\lambda_q = 0$. However, in this limit Δ and λ remain finite because the corresponding combinations of physical parameters are diverging. In particular we have

$$L = \frac{1}{N}\sum_{ij} m_i X_{ij} m_j \propto \frac{\Delta}{\lambda_q}. \tag{121}$$

Therefore L diverges at $\lambda_q = 0$. As we recalled in the introduction L is responsible for the behavior of the isolated eigenvalue and its divergence causes this eigenvalue to be zero. Thus in the space of the parameters (u, λ_q), conjugated respectively to the free energy and to the overlap, the extremization corresponds to set $u = 0$ and $\lambda_q = 0$, but the line $\lambda_q = 0$ is a singular line, and we must carefully take the limit. Instead the line $u = 0$ is regular and we can set $u = 0$ from the beginning as we did here. It would be interesting to check if the presence of two fields H_1 and H_2, instead of simply one, and the special role played by the self-overlap are connected to the fact that the equivalent replica theory requires the two-group ansatz [4, 11, 14, 32]. The formulation of the theory within the cavity method opens the way to applications to different models [6, 7] and optimization problems [8, 9]. Indeed the approach presented here has been successfully applied to recover the SUSY breaking complexity in diluted models [35, 36].

Within the SUSY framework we have shown that setting $\lambda_q \neq 0$ we break the SUSY of the problem thus removing the problem of the vanishing prefactor of the exponential contribution. However, the BM solution satisfies some SUSY Ward identities at any λ_q and this ensures its physical consistency with respect to the problems discussed in the introduction. The correct way of treating the $\lambda_q = 0$ case is by taking the limit $\lambda_q \to 0$, this is different from setting $\lambda_q = 0$ from the beginning that instead yields the incorrect standard SUSY identities. The SUSY Ward identities connect the bosonic order parameters of the theory with the fermionic ones. In this way they encode, within the standard computation of the TAP complexity, the physical meaning, obtained through the cavity method, of the bosonic order parameters in terms of the physical observables L, Z_1, Z_2 that are indeed related to averages of the macroscopic fermionic variables.

Although the results have been obtained within the SK model, they can be extended to all models where a BM-like solution for the total complexity exists, e.g. the Ising p-spin model [33]. In these models there is transition from a non-SUSY solution at high free energies to the SUSY solution [34] at low free energies [37]. Actually in all known FRSB and 1RSB models the complexity at the lower band edge, given by the Parisi solution, is SUSY; this could be related to the fact that in some sense the relevant parameter for the non-SUSY complexity is the self-overlap and not the free energy; instead at the lower band edge the free energy must be the relevant parameter since we want to recover thermodynamics.

Let us also note that it would be interesting to use the information obtained on the behaviour of the TAP solutions upon changes of the external parameters to gain further insight in the longstanding questions of chaos in temperature and magnetic field (see, e.g. ,discussion in [38]).

After the preparation of this manuscript the formulation of the BM theory within the cavity method has been rederived through a different approach [30]. The basic idea is that when a new spin is added the TAP equations for the remaining N spins are the TAP equations of the original system in the presence of non-zero magnetic fields $\{h_i\}$. This argument does not rely on any hypothesis on the possibility of continuing the solutions and can be used to derive a self-consistency equation for the probability $P(m_0|h_{0\,\mathrm{ext}})$ to find a given value of the magnetization m_0 conditioned to the presence of an external field $h_{0\,\mathrm{ext}}$ on it ($h_{0\,\mathrm{ext}}$ is called h_0 in [30]). This procedure leads to the BM equations for the parameters Δ, λ and q. Then one finds out that if the solution is SUSY one can write down a self-consistency equation for $P(m_0)$ alone while in order to derive the BM solution the full conditioned distribution $P(m_0|h_{0\,\mathrm{ext}})$ must be used. Therefore the essential feature of the BM solution within the cavity method is the same in both approaches: we cannot write down a self-consistency equation only for the distribution $P(H_1)$; instead, another fluctuating field must be present in the theory in such a way that we can study the self-consistency equation of the corresponding joined or conditioned distribution $P(H_1, H_2)$ or $P(m_0|h_{0\,\mathrm{ext}})$.

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References

[1] Mézard M, Parisi G and Virasoro M A 1987 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
[2] Bouchaud J-P, Cugliandolo L F, Kurchan J and Mezard M *Spin-Glasses and Random Fields* ed A P Young (Singapore: World Scientific)
[3] Yedidia J S, Freeman W T and Weiss Y 2001 Understanding belief propagation and its generalizations *MERL Technical Report*
Yedidia J S, Freeman W T and Weiss Y Generalized belief propagation *Advances in Neural Information Processing Systems* vol 13 (Cambridge, MA: MIT Press) pp 689–95
[4] Bray A J and Moore M A 1980 *J. Phys. C: Solid State Phys.* **13** L469
[5] Mézard M, Parisi G and Virasoro M A 1986 *Europhys. Lett.* **1** 77 reprinted in [1]
[6] Mézard M and Parisi G 2001 *Eur. Phys. J.* B **20** 217
[7] Mézard M and Parisi G 2003 *J. Stat. Phys* **111** 1
[8] Mezard M and Zecchina R 2002 *Phys. Rev.* E **66** 056126
[9] Mezard M, Parisi G and Zecchina R 2002 *Science* **297** 812
[10] Thouless D J, Anderson P W and Palmer R G 1977 *Phil. Mag.* **35** 593 reprinted in [1]
[11] Parisi G and Potters M 1995 *Europhys. Lett.* **32** 13
[12] Cavagna A, Giardina I, Parisi G and Mézard M 2003 *J. Phys. A: Math. Gen.* **36** 1175
[13] Annibale A, Cavagna A, Giardina I, Parisi G and Trevigne E 2003 *J. Phys. A: Math. Gen.* **36** 10937
Annibale A, Cavagna A, Giardina I and Parisi G 2003 *Phys. Rev.* E **68** 061103
[14] Crisanti A, Leuzzi L, Parisi G and Rizzo T 2003 *Phys. Rev.* B **68** 174401
[15] Crisanti A, Leuzzi L, Parisi G and Rizzo T 2004 *Phys. Rev.* B **70** 064423
Crisanti A, Leuzzi L, Parisi G and Rizzo T 2004 *Phys. Rev. Lett.* **92** 127203
[16] Parisi G and Sourlas N 1979 *Phys. Rev. Lett.* **43** 744
[17] Parisi G and Sourlas N 1982 *Nucl. Phys.* B **206** 321
[18] Becchi C, Stora A and Rouet R 1975 *Commun. Math. Phys.* **42** 127
Tyutin I V 1975 *Lebdev preprint* FIAN **39** unpublished
[19] Zinn-Justin J 1989 *Quantum Field Theory and Critical Phenomena* (Oxford: Clarendon)
[20] Kurchan J 1991 *J. Phys. A: Math. Gen.* **24** 4969–79
[21] Aspelmeier T, Bray A J and Moore M A 2004 *Phys. Rev. Lett.* **92** 087203
[22] Parisi G and Rizzo T 2004 *J. Phys. A: Math. Gen.* **37** 7979–92
[23] Bray A J and Moore M A 1979 *J. Phys. C: Solid State Phys.* **12** L441
[24] Plefka T 1982 *J. Phys. A: Math. Gen.* **15** 1971
Plefka T 2002 *Europhys. Lett.* **58** 892
[25] Plefka T 2002 *Phys. Rev.* B **65** 224206
[26] Mehta M L 1967 *Random Matrices* (New York: Academic)
[27] Bray A J, Moore M A and Young A P 1984 *J. Phys. C: Solid State Phys.* **17** L155
[28] Plefka T 2002 *Phys. Rev.* B **65** 224206
Plefka T 2003 *Preprint* cond-mat/0310782
[29] Cavagna A, Giardina I and Parisi G 2004 *Phys. Rev. Lett.* **92** 120603
[30] Cavagna A, Giardina I and Parisi G 2004 *Preprint* cond-mat/0407440
[31] Parisi G private communication
[32] Bray A J and Moore M A 1068 *Phys. Rev. Lett.* **41** 1978
[33] Rieger H 1992 *Phys. Rev.* B **46** 14655
[34] Montanari A and Ricci-Tersenghi F 2003 *Eur. Phys. J.* B **33** 339
[35] Rizzo T 2004 On the Complexity of the Bethe Lattice Spin-Glass *Preprint* cond-mat/0404729
[36] Parisi G and Rizzo T 2004 *Zero-Temperature Limit of the SUSY-breaking Complexity in Diluited Spin-Glass Models Preprint* cond-mat/0411732
[37] Crisanti A, Leuzzi L, Parisi G and Rizzo T 2004 *Preprint* cond-mat/0406649
[38] Rizzo T 2001 *J. Phys. A: Math. Gen.* **34** 5531–49
Rizzo T and Crisanti A 2003 *Phys. Rev. Lett.* **90** 137201